Sweepless Time-Dependent Transport Calculations using the Staggered Block Jacobi Method

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Abstract
The Staggered Block Jacobi method for time-dependent transport problems is a sweepless and inherently parallel transport method. It is highly accurate in thick-diffusive problems and unconditionally stable when combined with the lumped linear discontinuous finite element spatial discretization.

Introduction
The linear Boltzmann transport equation describes a rarified field of neutral particles streaming through and interacting with a background material.

The transport equation:
\[
\begin{align*}
\frac{1}{v} \frac{\partial}{\partial t} \phi + \mu (\psi - \phi) + \Sigma_q \phi + Q = 0
\end{align*}
\]

- Streaming Term
- Collision Term
- Source

The time-dependent transport equation is useful for a variety of problems including astrophysics simulations and inertial confinement fusion.

Classification of Time Discretizations
Currently, there were two major classifications of time discretizations:

- Conditionally stable discretizations:
  - Typically fast for a single time step
  - Often scale linearly with processors
  - Require very small \( \Delta t \) for stability
  - Examples include Explicit and Crank-Nicolson

- Unconditionally stable discretizations
  - Typically require non-linear solvers
  - Require sweeps, which limits parallel scalability
  - Example: Implicit

Mesh Sweeps
Traditionally deterministic transport problems have been solved using a mesh-sweep algorithm:

\[
\begin{align*}
&\text{Mesh-sweep}: 3 \rightarrow 2 \rightarrow 1 \rightarrow 3 \\
&\text{Mesh-sweep algorithm: Require that each cell be solved in a specific sequence. This limits the parallel efficiency of the transport algorithm.}
\end{align*}
\]

Additionally, transport methods employing mesh-sweeps typically iterate over the scattering source. This requires diffusion synthetic acceleration and Krylov solvers, further complicating the design, implementation, and parallelization of the transport method.

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Staggered Block Jacobi Method (SBJ)

Staggered Block Jacobi Concept: Invert a two-cell block (1-D) using incident information lagged to the previous time step. Only angular fluxes on the two-cell interface are retained.

This method should be accurate where the solution does not change greatly during a time step (i.e., where the wave speed is slow).

Increasing Accuracy with Iterations
We can improve the accuracy of the SBJ scheme using iterations along with sweeps:
1. Perform a stretched sweep to capture the solution in thin streaming regions
2. Perform a SBJ solve to capture the solution in thick diffusive regimes
3. Use the results from the SBJ as the incoming flux on each block
4. Repeat steps 2 and 3 until the desired accuracy is achieved.

Results
- Mesh width: 1.0 cm
- Number of zones: 100
- \( \Sigma_q = 10^{-7} \times 10^{-5} \)
- \( Q = 1.0 \) in first cell, \( Q = 0.0 \) elsewhere
- \( v = 1.0 \) cm/s

We calculate the relative error using:

Using Sweeps to Increase Accuracy

SBJ is accurate when the particle wave moves less than one cell per time step. Otherwise, accuracy degrades. We can improve accuracy using a single sweep with a lagged scattering source.

The term \( \frac{1}{c} (c = 1) \) is the inverse of the mean free path of the particle. We want to maximize the mfp, therefore

\[
\begin{align*}
\ell = 1.0 + \sqrt{\Delta t_{1+1}}
\end{align*}
\]

Using Sweeps to Increase Accuracy

Future Work
We are now applying this work to non-linear thermal radiation transport problems.