A Wave Model for One-Dimensional Gray Diffusion

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Wave Structure

Hyperbolic part of rad-hydro equations has 4-wave structure. In the optically-dense limit, system has quasi-linear form:

\[
U_1 + A(U)U_2 = S
\]

\[
U = (\rho, \rho u, E, E)_T^{\gamma} \text{ and } S = (0, 0, -S_{exch}, S_{exch})^T.
\]

Sound Speeds:

\[
\alpha^2 = \frac{2p}{\rho} \quad \text{and} \quad \beta^2 = \frac{(4/3)p_{rad}}{\rho}
\]

Wave Speeds:

\[
\lambda_1 = u - c_1 - c_2 \quad \text{(left acoustic)}
\]

\[
\lambda_2 = u \quad \text{(particle wave)}
\]

\[
\lambda_3 = u + 2c_1 \quad \text{(radiative particle wave)}
\]

\[
\lambda_4 = u - c_1 + c_2 \quad \text{(right acoustic)}
\]

with

\[
c_1 = \frac{1}{2}\sqrt{a^2 + b^2} \left(\frac{1}{\sqrt{3}} \cos(\theta/3) - \sin(\theta/3)\right)
\]

\[
c_2 = \frac{1}{2}\sqrt{a^2 + b^2} \left(\sqrt{3} \cos(\theta/3) + \sin(\theta/3)\right)
\]

\[
\theta = \arctan \left(\frac{\sqrt{4/27}(a^2 + b^2) - (\gamma - 1)u^2b^2}{(\gamma - 1)u|b|^2}\right)
\]

Eigenvectors:

\[
r_1 = \begin{pmatrix} \lambda_1 \\ Q_1 \\ 3\lambda_2 \end{pmatrix}, \quad k = 1, 3, 4 \\

r_2 = \begin{pmatrix} u \\ u^2/2 \\ 0 \end{pmatrix}
\]

with

\[
Q_1 = \frac{2M + 2u(\gamma - 3)\lambda_1 - (\gamma - 3)u^2 - 2b^2}{2(\gamma - 1)}
\]

Source: Parameterize \(S_{exch}\) by

\[
S_{exch} = R \left(\frac{p}{\rho}\right) - AE_{rad}
\]

\(R, A\): Radiative and absorptive parameters

Shock Tube Data

Sod’s shock tube with uniform radiation energy:

\[
(\rho, u, p, p_{rad}, \gamma) = \begin{cases} (1, 0.1, 0.1, 5/3), & x < 0 \\ (1/8, 0.01, 0.1, 5/3), & x > 0 \end{cases}
\]

Source parameters: \(R = A = 1\)

Contact Data

Across a contact: \(p + p_{rad} = P_0\), \(u = U_0 \implies (P_0 + U_0(p_{rad})_x = (\gamma - 1)U_0(p_{rad})_x + \frac{1}{2} - \gamma) S_{exch}\)

Total pressure \(P_0\) is constant if matter pressure satisfies an ODE:

\[
(\gamma - 1)U_0p_{\rho_x} = \frac{1}{2} S_{exch} \quad p(x_L) = p_L
\]

NOTE: Individual pressures are not constant unless \(\gamma = 4/3\) (photons and matter are “same fluid”)

Source Effects

- Four waves identified in density profile
- Sum of pressures constant across particle wave
- Particle waves clustered together near contact

Contact

- Large radiative term slows acoustic waves and speeds up radiative particle wave
- If \(R/A \gg 1\), radiative wave collides with acoustic
- Particle waves close together when \(R\) is small. Difficult to resolve wave structure.