Dynamical Cores: Design and the Designer

Richard B. Rood
rbrood@umich.edu
Atmospheric, Oceanic and Space Sciences
University of Michigan

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Some References (Lin & Rood)

Plan of Presentation

• **Introduction**
  – Models
  – Definition of dynamical core

• **Governing equations: What do they tell us?**
  – Momentum equations
  – Constituent equations

• **Interface of the governing equations with numerical approximation**
  – Motivators
  – Design

• **Impact on geophysical performance of model**

• **Spectra and gravity waves: My confusions**

• **Conclusions**
Model and Modeling

• Model
  – A work or construction used in testing or perfecting a final product.
  – A schematic description of a system, theory, or phenomenon that accounts for its known or inferred properties and may be used for further studies of its characteristics.

Types: Conceptual, Statistical, Physical, Mechanistic, …
Big models contain little models

Management of complexity
But, complex and costly

Where are chemistry and aerosols?
Organizing models

\[ \frac{\partial A}{\partial t} = - \nabla \cdot \mathbf{U} A + M + P - LA \]

\[ \frac{\partial A}{\partial t} = \text{Dynamical Core} + \text{Physics} \]
Remember how we got here.

Reminder: There are dynamical cores for oceans as well. Oh, and sea ice advects, and ground water flows through the Earth. Who knows what’s happening in ice sheets?
What’s in a dynamical core?

Dynamical Core = Horizontal “Resolved” Advection

+ Vertical “Resolved” Advection

+ “Unresolved” Subscale Transport

+ Filters and fixers

And a way to advance all of these pieces in time.
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• **Conclusions**
Equations of motion

• Let’s think about the equations of motions and the fact we are going to try to approximate them for the real Earth.

• We are faced with mountains and land and oceans and trees and water and ice and clouds and water and sand and ozone and pavement and water and, ultimately, humility.
Equations of motion
(z, height, as vertical coordinate)

\[
\frac{Du}{Dt} - \frac{uv\tan(\phi)}{a} + \frac{uw}{a} = \frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin(\phi) - 2\Omega w \cos(\phi) + \nabla^2 (u) \tag{Eq. 1}
\]

\[
\frac{Dv}{Dt} + \frac{u^2 \tan(\phi)}{a} + \frac{vw}{a} = \frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin(\phi) + \nabla^2 (v) \tag{Eq. 2}
\]

\[
\frac{Dw}{Dt} - \frac{u^2 + v^2}{a} = \frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2\Omega u \cos(\phi) + \nabla^2 (w) \tag{Eq. 3}
\]

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \tag{Eq. 4}
\]

\[
c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = J \quad \text{or} \quad \frac{c_p}{T} \frac{DT}{Dt} - \frac{R}{P} \frac{Dp}{Dt} = \frac{J}{T} \tag{Eq. 5}
\]

\[
p = \rho RT \quad \text{and} \quad \alpha = \frac{1}{\rho} \tag{Eq. 6}
\]

**tangential coordinate system on Earth’s surface**
(x, y, z) = (+ east, + north, + local vertical)
Comment: Horizontal Pressure Gradient

\[
\begin{align*}
\frac{Du}{Dt} - \frac{uv\tan(\phi)}{a} + \frac{uw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin(\phi) - 2\Omega w \cos(\phi) + v\nabla^2(u) \quad \text{(Eq. 1)} \\
\frac{Dv}{Dt} + \frac{u^2\tan(\phi)}{a} + \frac{vw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin(\phi) + v\nabla^2(v) \quad \text{(Eq. 2)}
\end{align*}
\]

Initiator of motion. If there are numerical errors here, then it is an initiator of erroneous motion. How you treat the vertical coordinate, especially topography, has a big impact here. Lin, S. J., 1997:

Tangential coordinate system on Earth’s surface \((x, y, z) = (+ \text{ east, } + \text{ north, } + \text{ local vertical})\)
Comment: Dissipation

There is, formally, dissipation in the equations of motion. This dissipation becomes entangled, wound up, indeed, conflated with dissipation that is in the numerical approximation for bad and good reasons.

\[ \frac{Du}{Dt} + \frac{uv \tan(\phi)}{a} + \frac{uw}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin(\phi) - 2\Omega w \cos(\phi) + v \nabla^2(u) \]  

(Eq. 1)

\[ \frac{Dv}{Dt} + \frac{u^2 \tan(\phi)}{a} + \frac{vw}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin(\phi) + v \nabla^2(v) \]  

(Eq. 2)

\[ \frac{Dw}{Dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2\Omega u \cos(\phi) + v \nabla^2(w) \]  

(Eq. 3)

\[ c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = J \text{ or } \frac{c_p}{T} \frac{DT}{Dt} - \frac{R}{P} \frac{Dp}{Dt} = \frac{J}{T} \]  

(Eq. 5)

tangential coordinate system on Earth’s surface
\( (x, y, z) = (+ \text{ east, } + \text{ north, } + \text{ local vertical}) \)
There is, explicitly, dissipation in the parameterizations that we include in the models. This dissipation becomes entangled, wound up, indeed, conflated with dissipation that is in the numerical approximation for bad and good reasons.
Comment: Dissipation

Dynamical Core = Horizontal “Resolved” Advection + Vertical “Resolved” Advection + “Unresolved” Subscale Transport + Filters and fixers

Dynamics / Physics:
- Advection
- Mixing
- Pl_Boud_Layer
- Gravity Waves
- Radiation
- Convection
- Clouds

We should expect trouble.

We have sensitivity to Resolution Velocity
We really must have some constituent equations.

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \quad \text{(Eq. 4)}
\]

\[
\frac{D\rho_{H_2O}}{Dt} = -\rho_{H_2O} \nabla \cdot \mathbf{u} + P_{H_2O} - L_{H_2O} \quad \text{(Eq. 7)}
\]

\[
\frac{D\rho_{O_3}}{Dt} = -\rho_{O_3} \nabla \cdot \mathbf{u} + P_{O_3} - L_{O_3} \quad \text{(Eq. 8)}
\]

\[
\frac{D\rho_{\text{Who\_Knows?}}}{Dt} = -\rho_{\text{Who\_Knows?}} \nabla \cdot \mathbf{u} + P_{\text{Who\_Knows?}} - L_{\text{Who\_Knows?}} \quad \text{(Eq. 9 - Who\_knows?)}
\]

There is a lot of “small-scale” structure that comes into the approximation of the Earth’s climate when we consider the role of trace constituents.

Small scale structure comes from:
- distribution of sources and sinks
- dynamical regimes
- hyperbolic nature of the equations
- advective cascade, shear, ....
Comment: Constituents (scalars)

• When we consider constituents we are brought into harsh confrontation with small-scale structure. This should, after more than a week of lectures, immediately send off an alarm about ripples, overshoots and undershoots.

• Practically, productions and loss algorithms hate overshoots and undershoots, especially those negative densities.
Comment: Constituents (scalars)

- From first principles: overshoots and undershoots generated by a numerical scheme are not physical. Let me say that again, they are not physical.
- The numerical scheme can, therefore, act as a source or sink term.
  - And there are many other ways that this can be conceptually framed.
Comments: Constituents ( Scalars )

• Constituents provide a powerful way to study dynamics.
  – Follow their path through the atmosphere:
    • Advection versus diffusion
  – Their ultimate distribution, their “well-mixed” state, is a function of the accumulated circulation, \textit{i.e.} transport, and production and loss time scales.

• A strategy that has evolved is to treat tracers (and temperature) with a different numerical method from momentum.
  – A personal belief: I prefer to avoid this.
Comments: Pulling-it together

- The dynamics of the atmosphere are “hyperbolic,” which means that they can generate fronts.
- There are physical, dissipative processes in the atmosphere.
- There are dynamical processes which we simulate as dissipative.
- Dynamical cores, modeling advection, introduce dissipative and dispersive errors.
  - How the time scales of these algorithmic errors stack up against the time scales that come from the physical time scales in the climate system is an important consideration.
  - This will depend on the specifics of the application which you are trying to simulate.
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• Interface of the governing equations with numerical approximation
  – Motivators
  – Design

• Impact on geophysical performance of model
• Spectra and gravity waves: My confusions
• Conclusions
Two common paths of development

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<td>Partial Differential Equations</td>
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<td>Numerical Approximation</td>
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<tr>
<td>Intuitive representation of physics</td>
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<tr>
<td>Numerical Approximation</td>
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</table>

Should be, but we are faced with the real world of numerical approximation and discrete mathematics
Some motivation from constituents
Consider the measurements of atmospheric constituents

Enormous amount of information:
- Mixing physics
- Mixing time scales
- Chemical production and loss

These are observations made by airplanes. They are, for all practical aspects, instantaneous, point measurements.
Spectral method and correlations

Sources of pathology
- Inability to fit local features
- Inconsistency between tracer and fluid continuity equation
- Dispersion errors
- Filtering

Spectral Method (widens over time)
Van Leer method and correlations

Why does this work?
• Consideration of volumes and mixing these volumes consistently.
Some basic problems of numerical advection

from Rood, Rev Geophys, 1987

There is a tension between:

“Dispersion” errors
And

“Diffusion” errors

Fundamental to linear schemes. Leads to development of non-linear schemes.
Design features for resolved advection


• Conserve mass without *a posteriori* restoration
• Compute fluxes based on the subgrid distribution in the upwind direction
• Generate no new maxima or minima
• Preserve tracer correlations
• Be computationally efficient in spherical geometry
Design Features

• Machenhauer et al. (2008) for a more complete discussion of design features.
With the ultimate goal

\[ \frac{\partial A}{\partial t} = \text{Dynamical Core} + \text{Physics} \]

Model resolved advection accurately

Model sub-scale transport credibly

Minimize filters and fixes

Provide to the physics parameterizations estimates that are physically realizable

Leave no artifact that stops observationally based model evaluation
So what is the Lin-Rood scheme for horizontal “resolved” advection?

- It’s a prescription of how to build a model

Define: \( Q = \pi q \)

\[
F \left( u^*, \Delta t, \Delta x; Q^n \right) = -DIF \left[ \chi \left( u^*, \Delta t, \Delta x; Q^n \right) \right]
\]
Split $x$ and $y$ operations

- An operator in $x$ direction and one in $y$ direction

$$F (u^*, \Delta t, \Delta x; Q^n) = -DIF [X (u^*, \Delta t, \Delta x; Q^n)]$$

$$G (v^*, \Delta t, \Delta y; Q^n) = -DIF [Y (v^*, \Delta t, \Delta y; Q^n)]$$
Take a step in the x direction

\[ Q^x = Q^n + F(Q^n) \]

• Followed by a step in the y direction

\[ Q^{yx} = Q^x + G(Q^x) \]

• On substitution

\[ Q^{yx} = Q^n + F(Q^n) + G(Q^n) + GF(Q^n) \]
Take a step in the y direction

\[ Q^y = Q^n + G(Q^n) \]

- Followed by a step in the x direction

\[ Q^{xy} = Q^y + F(Q^y) \]

- On substitution

\[ Q^{xy} = Q^n + F(Q^n) + G(Q^n) + FG(Q^n) \]
Take both and average

\[ Q^* = \frac{(Q^{xy} + Q^{yx})}{2} \]

\[ Q^* = Q^n + F(Q^n) + G(Q^n) + \frac{(Fg(Q^n) + Gf(Q^n))}{2} \]

• And do some art

This is a conventional approach, but it does NOT adhere to our rule, and hence does NOT meet our design criteria. What we (really SJ) noticed was if in the cross terms the inner operator was replaced with the advective form (with a special average velocity), \( f \) and \( g \), then we did meet these criteria.

\[ Q^* = Q^n + F(Q^n) + G(Q^n) + \frac{(Fg(Q^n) + Gf(Q^n))}{2} \]
And just to be clear

• You can chose your differencing algorithm.
  – We always chose schemes of the van Leer family

• There is more art in
  – Definition of time-averaged velocities
  – Placement of quantities on grid to assure consistency between potential vorticity and geopotential and tracers
    • C and D grids are both used
And with a lot omitted details

Conventional finite difference

Lin-Rood built with Piecewise Parabolic Method
Some comments: Lin-Rood

• There are a lot of schemes that can make nice pictures, like the previous viewgraph.
• Physically realizable estimates of modeled parameters
• There are more accurate schemes. (Prather’s Advection Scheme)
• There is diffusion. (This is a design choice.)
  – It is scale dependent
  – It is resolution dependent
  – It seems to be well behaved with resolution (Really?)
• There are some issues of non-linearity
  – We made some linearity assumptions along the way
• There are some issues of splitting
  – Issues of orthogonality in splitting
• There are some issues of stability
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Some papers that document the changes (generally improvements) in representing geophysical phenomena

• **Douglass AR, Schoeberl MR, Rood RB, et al., 2003**: Evaluation of transport in the lower tropical stratosphere in a global chemistry and transport model

• **Schoeberl MR, Douglass AR, Zhu ZX, et al., 2003**: A comparison of the lower stratospheric age spectra derived from a general circulation model and two data assimilation systems

• **Atlas R, Reale O, Shen BW, et al., 2005**: Hurricane forecasting with the high-resolution NASA finite volume general circulation model
Some papers that document the changes (generally improvements) in representing geophysical phenomena:

- **Bala, G; Rood, RB; Mirin, A, et al., 2008:** Evaluation of a CCSM3 simulation with a finite volume dynamical core for the atmosphere at 1 degrees latitude x 1.25 degrees longitude resolution. *Journal of Climate* 21 (7) Pages: 1467-1486.

Comments: Drawn from applications

- Lagrangian vertical coordinate led to fundamental improvement in the general circulation, specifically, the meridional mass circulation.
- Consistency between constituents and potential vorticity central to ability to determine cause and effect in mechanistic studies.
- Locality of scheme leads to improvements in the geophysical robustness near steep topography and coastlines.
  - Consistency means: budgets are not strongly impacted by filters and fixers.
- Dissipation and diffusivity requires further study and is, at times, confounding.
- Behavior as resolution increases is not as simple as originally projected.
Looking at locality: Water, water, water

• One on my original motivations for Lin-Rood type dynamics was
  – What does it do to water and precipitation and clouds and water and water and radiation and water.
Part of my motivation

(Iorio et al., Climate Dynamics, 2004)

Spectral Model
High resolution

Western U.S. Precipitation

T42 ~ 4 degrees
T85 ~ 2 degrees
T170 ~ 1 degree
T239 ~ .75 degree

Relative insensitivity of western U.S. precipitation to resolution. Note separation of coastal ranges and high Rockies in Northwest.
Comparison of models and resolutions
(improvement of patterns)

T85 Spectral

T42 ~ 4 degrees
T85 ~ 2 degrees
T170 ~ 1 degree
T239 ~ .75 degree

Color scales are different!

T170 Spectral

Observations

1 degree LR

0.5 degree LR
Closer look (0.5 degree LR)

“Pixels” rather than contours to show contrast of wet and dry areas.
Clouds and short wave cloud forcing

Clouds: Spectral Ringing

Radiative Forcing: Spectral Ringing

Bala, G; Rood, RB; Bader, D., et al., 2008:
Clouds and short wave cloud forcing

Radiative Forcing: For Reference (W/M²)
- 80  -64  -144

Bala, G; Rood, RB; Bader, D., et al., 2008:
Locality: Pulling it together
(from Bala et al., 2008)

- Precipitation
  - More realistic distribution of rain, wet regions and dry regions, near topography.
  - Better representation of local circulations which organize precipitation.
  - Does not address fundamental issues of convective parameterizations.
- Ocean surface stress near steep topography
  - Flow aligns more realistically with topography
  - Consistent changes in oceanic upwelling
  - Changes in sea surface temperature
- Clouds and radiation near steep topography
  - Reduction of spectral ringing in clouds
  - Changes in both instantaneous and average regional radiative characteristics
- Representation of topography should be part of the dynamical core.
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Saying something about spectra
(see Skamarock, 2004)

Baroclinic wave test @ 850mb from Dave Williamson

I’ve worried about spectra for years, but I have some problem figuring out what they tell me.

I’ve seen this argued as a representation of gravity waves, even when it was obviously a numerical artifact..

This steep fall off is characteristic of monotone, diffused schemes.
Comment: Dissipation

Dynamical Core = Horizontal “Resolved” Advection + Vertical “Resolved” Advection + “Unresolved” Subscale Transport + Filters and fixers

There is so much going on down here that I don’t find that spectra tell me much.
Comment: Dissipation

There is a lot going on at the high wave-number end of the spectrum

- Advective cascade
- Turbulence
- Parameterization
- Filters and fixers
- Resolved gravity waves

I have, depending on application, gone from over-diffused to under-diffused as a function of resolution.
  - horizontal and vertical mixing behave differently

I have never felt that I have modeled mixing in either a robust or extensible way.
Gravity waves

- Weather forecasting
  - Gravity waves are noise that need to be removed.
    - Old strategies: Numerical diffusion and filtering
  - Gravity wave drag decelerates the jet stream
    - Parameterization

- Climate modeling
  - Gravity waves decelerate the zonal wind above the stratosphere, and through “induced” circulation, warm the winter poles.
  - They are central to the general circulation and trace constituent distributions.
Gravity waves

• Weather forecasting (what I think I have learned from Bill Skamarock)
  – At high resolution the gravity waves are important dynamics of squall lines and mesoscale convective complexes ... they are not, first and foremost, diffusive!

• Climate modeling
  – Squall lines and mesoscale convective complexes are important to continental water budgets, etc.
High wave numbers

- We have some serious thinking to do about the high wave number end of the spectra. It is not simply a matter of the spectra.

- Courtesy and with permission of S.-J. Lin. This is fv cubic sphere (GFDL) ~ 0.5 degree. Experiment in progress. With explicit damping to curl(u).

- Proven to be a strong function of grid.

Generalized Held–Suarez forcing with USGS terrain
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Conclusions

• If you object to cooking animals, or specifically pork, cover your ears and hide your eyes.

• If you are offended by someone from Carolina wearing a cheap Duke sweat shirt, hide your eyes.
  – (Or if you think that red is the color of a school named Carolina.)

• If you are concerned about maintaining impeccable professional decorum to denote the seriousness of the subject – well, it’s too late.
Far more widely read than Lin and Rood is *Cape Anne Pig and Me*.

There are in the world thousands, if not millions, of cookbooks. Many of these books lay out exactly how, for example, a pork shoulder must be cooked. They maintain that if the pork shoulder is not treated by their prescription, many embarrassing and distasteful consequences will be realized. Many of these books directly conflict with others, and most produce a good pork shoulder. The lesson: people have been around a long time and getting things cooked by whatever means they have in front of them. It works out.
Conclusions

- Dynamical core has multiple components
  - Resolved and unresolved transport are both important.
    - Unresolved transport is tremendously important to climate
      - Reliance on *a posteriori* filters and fixes is, based on experience, bad.
- If a dynamical core does not represent the foundational physics evident in the observational record, then comparisons with observations are of limited analytical value.
- What is occurring in the scales less than 100 km is important, always has been, but as we model in this resolution we require rigorous investigation of dynamical cores and physical parameterizations as a system.
Conclusions

• If I were starting something new today
  – A “Lagrangian” vertical co-ordinate.
  – A quasi-equal-area, quasi-orthogonal grid
  – An advection scheme that is local, and upstream biased
  – A “consistent” representation of topography

• If I were going to have a research problem in dynamical cores, I would focus on deconstruction of the mixing and filters and fixers and their impact on simulations. (Wait, Christiane and I got funded. We are doing this.)

• If I were going to try to distract a graduate student with what might be a good or bad idea I would have then sit down for a month with the literature and my colleagues in the engineering departments and look at the formalization of multi-scale methods.
Earth System
What are models used for?

• Diagnostic: The model is used to test the processes that are thought to describe the observations.
  – Are processes adequately described?

• Prognostic: The model is used to make a prediction.
  – Deterministic
  – Probabilistic
# Simulation Environment

(General Circulation Model, “Forecast”)

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<tr>
<th>Boundary Conditions</th>
<th>Emissions, SST, …</th>
<th>( \varepsilon )</th>
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</thead>
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<tr>
<td>Representative Equations</td>
<td>( DA/Dt = M + P - LA )</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>Discrete/Parameterize</td>
<td>( (A_{n+\Delta t} - A_n)/\Delta t = \ldots )</td>
<td>( (\varepsilon_d, \varepsilon_p) )</td>
</tr>
<tr>
<td>Theory/Constraints</td>
<td>( \partial u_g/\partial z = -(\partial T/\partial y)R/(Hf_0) )</td>
<td>Scale Analysis</td>
</tr>
<tr>
<td>Primary Products (\textit{i.e.} A)</td>
<td>T, u, v, ( \Phi ), H(_2)O, O(_3) \ldots</td>
<td>( (\varepsilon_b, \varepsilon_v) )</td>
</tr>
<tr>
<td>Derived Products (F(A))</td>
<td>Pot. Vorticity, v*, w*, \ldots</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

\((\varepsilon_b, \varepsilon_v) = \text{(bias error, variability error)}\)

Derived Products likely to be physically consistent, but to have significant errors. \textit{i.e.} The theory-based constraints are met.
Representative Equations

• \( \frac{\partial A}{\partial t} = - \nabla \cdot \mathbf{U} A + M + P - LA \)
  
  – \( A \) is some conserved quantity
  – \( \mathbf{U} \) is velocity \( \rightarrow \) “resolved” transport, “advection”
  – \( M \) is “Mixing” \( \rightarrow \) “unresolved” transport, parameterization
  – \( P \) is production
  – \( L \) is loss

• All terms are potentially important – answer is a “balance”
Strategies for the discretization of the horizontal “resolved” advection (that I have used)

- Finite difference
- Square root scheme (Schneider, 1984)
- Spectral schemes
- Van Leer type schemes
- Lin-Rood

- Why did I go through this progression?
Why we quit using them

• Finite difference
  – Poles, cross polar flow, failure to represent dynamics near the pole, dispersion errors

• Square root scheme (Schneider, 1984)
  – Same as finite difference, plus non-physical, unstable in convergent flow

• Spectral schemes
  – Global representation of local features, dispersion errors, Gibb’s phenomenon, need for filtering

• Where we ended up
  – Van Leer type schemes
    • Diffusive, splitting, consistency between scalars and fluid continuity
  – Lin-Rood
    • Designed from experience with van Leer schemes
Lin-Rood: Some details
“Finite-difference” vs. “finite-volume”

- Finite-difference methods “discretize” the partial differential equations via Taylor series expansion – pay little or no attention to the underlying physics

- Finite-volume methods can be used to “describe” directly the “physical conservation laws” for the control volumes or, equivalently, to solve the integral form of the equations using the following 3 integral theorems:

  1. Divergence theorem: for the advection-transport process
  2. Green’s theorem: for computing the pressure gradient forces
  3. Stokes theorem: for computing the finite-volume mean vorticity using “circulation” around the volume (cell)

Lin and Rood (1996 (MWR), 1997 (QJRMS)), Lin (1997 (QJRMS), 2004 (MWR))
Let's consider the conservation equation with no production or loss.

1. \( \frac{\partial A}{\partial t} + \nabla \bullet UA = 0 \)

2. \( A = \pi q \)
   - Where \( q \) is a “mixing-ratio” like quantity
   - Where \( \pi \) is a “density” like quantity

3. \( \frac{\partial \pi q}{\partial t} + \nabla \bullet U\pi q = 0 \)
   - \( \pi \frac{\partial q}{\partial t} + \pi U \bullet \nabla q + q \frac{\partial \pi}{\partial t} + q \nabla \bullet U\pi = 0 \)
   - But, \( \frac{\partial \pi}{\partial t} + \nabla \bullet U\pi = 0 \) (conservation of mass)
   - So, \( \frac{\partial q}{\partial t} + U \bullet \nabla q = 0 \)
Rules of algorithm development

- \[ \frac{\partial \pi}{\partial t} + \nabla \cdot \mathbf{U}\pi = 0 \]
  - Mass continuity equation

- \[ \frac{\partial \pi q}{\partial t} + \nabla \cdot \mathbf{U}\pi q = 0 \]
  - Global integral of \( \pi q \) is exactly conserved
  - If \( q \) is spatially uniform, then discrete form degenerates to mass continuity equation or \( \nabla \cdot \mathbf{U} = 0 \) for incompressible flow (this connects the equations, consistency!)

- \[ \frac{\partial q}{\partial t} + \mathbf{U} \cdot \nabla q = 0 \]
  - Advective form of scalar conservation

- Plus a rule on computational efficiency
So what is the Lin-Rood scheme for horizontal “resolved” advection?

- It’s a prescription of how to build a model

Define: $Q = \pi q$

$$F \left( u^*, \Delta t, \Delta x; Q^n \right) = -DIF \left[ X \left( u^*, \Delta t, \Delta x; Q^n \right) \right]$$
Split x and y operations

- An operator in x direction and one in y direction

\[
F (u^*, \Delta t, \Delta x; Q^n) = -DIF[X (u^*, \Delta t, \Delta x; Q^n)]
\]

\[
G (v^*, \Delta t, \Delta y; Q^n) = -DIF[Y (v^*, \Delta t, \Delta y; Q^n)]
\]
Take a step in the x direction

\[ Q^x = Q^n + F(Q^n) \]

• Followed by a step in the y direction

\[ Q^{yx} = Q^x + G(Q^x) \]

• On substitution

\[ Q^{yx} = Q^n + F(Q^n) + G(Q^n) + GF(Q^n) \]
Take a step in the $y$ direction

$$Q^y = Q^n + G(Q^n)$$

• Followed by a step in the $x$ direction

$$Q^{xy} = Q^y + F(Q^y)$$

• On substitution

$$Q^{xy} = Q^n + F(Q^n) + G(Q^n) + FG(Q^n)$$
Take both and average

\[ Q^* = \frac{(Q_{xy} + Q_{yx})}{2} \]

\[ Q^* = Q^n + F(Q^n) + G(Q^n) \\
    + \frac{(Fg(Q^n) + Gf(Q^n))}{2} \]

• And do some art

This is a conventional approach, but it does NOT adhere to our rule, and hence does NOT meet our design criteria. What we (really SJ) noticed was if in the cross terms the inner operator was replaced with the advective form (with a special average velocity), f and g, then we did meet these criteria.

\[ Q^* = Q^n + F(Q^n) + G(Q^n) \\
    + \frac{(Fg(Q^n) + Gf(Q^n))}{2} \]
Discretization of Resolved Transport

- $\frac{\partial A}{\partial t} = - \nabla \cdot UA$

**Gridded Approach**
- Orthogonal?
- Uniform area?
- Adaptive?
- Unstructured?

Choice of where to Represent Information

Choice of technique to approximate operations in representative equations

Rood (1987, Rev. Geophys.)
Discretization of Resolved Transport
Discretization of Resolved Transport

Choice of where to Represent Information Impacts Physics
• Conservation
• Scale Analysis Limits
• Stability
Discretization of Resolved Transport

• $\frac{\partial Q}{\partial t} = - \nabla \cdot U Q = - (\frac{\partial u^* Q}{\partial x} + \frac{\partial v^* Q}{\partial y})$
Discretization of Resolved Transport

• $\frac{\partial A}{\partial t} = - \nabla \cdot \mathbf{U} A$

Line Integral around discrete volume
The importance of your decisions

1-D "Tape Recorder" model (80L)

Year-1

Year-3

Year-6

Pressure (mb)

LCV: Lagrangian Control-Volume; SLT: Semi-Lagrangian Transport; CD2: 2nd order Center Diff
Importance of your decisions
(Tape recorder in full Goddard GCM circa 2000)

FINITE-VOLUME
Slower ascent
Faster mean vertical velocity

FINITE-DIFFERENCE
Faster ascent
Slower mean vertical velocity

S. Pawson, primary contact
More on Precipitation
Importance of your decisions
(Precipitation in full GCM)

T239 (50 km)  0.4° x 0.5° (40 x 50 km)  Observations (VE)

Spectral Dynamics Community Atmosphere Model / “Eulerian”

Finite Volume Dynamics Community Atmosphere Model / “Finite Volume”

Precipitation in California (from P. Duffy)
Mexican / S.W. U.S. Monsoon

Observations of Precipitation
(total mm for July, average)

Note extension in northern Mexican, and southern Arizona and New Mexico

Note Maximum

Note strong link to coastal mountains

FIG. 4. July mean precipitation (mm) from WMO (1975) atlas. Continental divide is marked by hatched line.
**T85 Spectral**

Fig. 4. July mean precipitation (mm) from WMO (1975) atlas. Continental divide is marked by hatched line.
1 degree Lin-Rood

Fig. 4. July mean precipitation (mm) from WMO (1975) atlas. Continental divide is marked by hatched line.
0.5 degree Lin-Rood

Fig. 4. July mean precipitation (mm) from WMO (1975) atlas. Continental divide is marked by hatched line.
Over South America
T85 – 1 degree LR

Changes in oceanic rain
Over South America
T85 – 0.5 degree LR

Changes in oceanic rain
Some conclusions about modeling

• Physical approach versus a mathematical approach
  – Pay attention to the underlying physics – seek physical consistency
  – How does my comprehensive model relate to the heuristic models?

• Quantitative analysis of models and observations is much more difficult than ‘building a new model.’ This is where progress will be made.
  – Avoid coffee table / landscape comparisons